WAVE REFLECTION AND TRANSMISSION IN ISOTROPIC MEDIA

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Abstract

The amplitude and power ratios for transmitted and reflected mechanical waves at an interface are derived for the simplest case of isotropic media. A comparison is drawn with the derivation of Fresnel's equations for light and the analogous meaning of the continuity of gradient of electric field components tangential to the interface.

Introduction

When a wave reaches an interface (change in refractive index), reflection and refraction take place. The simplest case considered here concerns isotropic media and a distinct uniform interface. The amplitude ratios, also termed reflection coefficient and transmission coefficient, and hence also the fractional energies of waves can thus be derived from boundary considerations, with knowledge of the law of reflection and Snell's law. In the case of electromagnetic waves, i.e. light, the results for amplitude ratios for light are the Fresnel equations. The boundary conditions for the derivation of Fresnel's equations arise from Maxwell's equations, in contrast to those for mechanical waves used here.

Derivation of amplitude ratios

Let **k** be the propagation vector of magnitude $k = 2\pi/\lambda$ and with direction being that of the direction of propagation of the wave, **r** be the position vector and *A* be the amplitude of the wave. The subscripts i,r,t denote incident, reflected, and transmitted respectively.
η_i is the refractive index of the medium of the incident (and also reflected) wave, and η_t is the refractive index of the medium of the transmitted wave, such that the refractive indices are inversely proportional to the wave speeds in the media and thus $k_i/k_t = \eta_i/\eta_t$.
Let the wave function of the incident wave ϕ_i (some function of space and time) be

$$\phi_i = A_i \exp[i(\mathbf{k_i}\cdot\mathbf{r} - wt)] \qquad (1)$$

Knowing also that frequency is constant, similar expressions can be written for the reflected and transmitted waves:

$$\phi_r = A_r \exp[i(\mathbf{k_r}\cdot\mathbf{r} - wt)] \qquad (2)$$

$$\phi_t = A_t \exp[i(\mathbf{k_t}\cdot\mathbf{r} - wt)] \qquad (3)$$

Let the boundary point where the waves meet on the interface be $\mathbf{r} = \mathbf{0}$.
Considering the boundary conditions, the boundary displacement should be continuous, i.e. displacement on immediate either side of the boundary should be equal. In other words, the displacement of the boundary as given by the superpositions of the incident and reflected waves should equal. The boundary gradient with respect to space should also be continuous, because if not then at the point of discontinuity there will be a finite force on an infinitesimal portion of

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medium, i.e. infinite acceleration. These boundary conditions should always be true regardless of time, i.e. they hold at **r** = **0** and for all t.

Thus for the continuity of displacement at the boundary:

$$\phi_i + \phi_r = \phi_t \tag{4}$$

Putting **r** = **0** :

$$A_i \exp[i(-wt)] + A_r \exp[i(-wt)] = A_t \exp[i(-wt)] \tag{5}$$

Since this holds for all time t,

$$A_i + A_r = A_t \tag{6}$$

For the continuity of gradient at the boundary:

$$\nabla\phi_i + \nabla\phi_r = \nabla\phi_t \tag{7}$$

Using the subscript j for cartesian components, such that $\hat{e}_j$ is the unit vector along x_j and

k = $\sum_j \hat{e}_j k_{Xj}$ and **r** = $\sum_j \hat{e}_j x_j$, we can also write

$$\sum_j \hat{e}_j \frac{\partial\phi_i}{\partial x_j} + \sum_j \hat{e}_j \frac{\partial\phi_r}{\partial x_j} = \sum_j \hat{e}_j \frac{\partial\phi_t}{\partial x_j} \tag{8}$$

$$A_i \exp[i(\mathbf{k_i}\cdot\mathbf{r} - wt)]\sum_j \hat{e}_j \frac{\partial}{\partial x_j} i(\mathbf{k_i}\cdot\mathbf{r} - wt) + A_r \exp[i(\mathbf{k_r}\cdot\mathbf{r} - wt)]\sum_j \hat{e}_j \frac{\partial}{\partial x_j} i(\mathbf{k_r}\cdot\mathbf{r} - wt)$$

$$= A_t \exp[i(\mathbf{k_t}\cdot\mathbf{r} - wt)]\sum_j \hat{e}_j \frac{\partial}{\partial x_j} i(\mathbf{k_t}\cdot\mathbf{r} - wt) \tag{9}$$

but $$\frac{\partial}{\partial x_j}(i(\mathbf{k}\cdot\mathbf{r} - wt) = \frac{\partial}{\partial x_j}(i\sum_j (k_{Xj} x_j) = ik_{Xj},$$

therefore $$\sum_j \hat{e}_j \frac{\partial}{\partial x_j}(i(\mathbf{k}\cdot\mathbf{r} - wt) = \sum_j \hat{e}_j\, ik_{Xj} = i\mathbf{k}$$

So (9) becomes

$$A_i \exp[i(\mathbf{k_i}\cdot\mathbf{r} - wt)]i\mathbf{k_i} + A_r \exp[i(\mathbf{k_r}\cdot\mathbf{r} - wt)]i\mathbf{k_r} = A_t \exp[i(\mathbf{k_t}\cdot\mathbf{r} - wt)]i\mathbf{k_t} \tag{10}$$

Putting **r** = **0** :

$$A_i \exp[i(-wt)]i\mathbf{k_i} + A_r \exp[i(-wt)]i\mathbf{k_r} = A_t \exp[i(-wt)]i\mathbf{k_t} \tag{11}$$

The boundary condition holds for all t, therefore

$$A_i\mathbf{k_i} + A_r\mathbf{k_r} = A_t\mathbf{k_t} \tag{12}$$

From equation (12), $\mathbf{k_i}$, $\mathbf{k_r}$, and $\mathbf{k_t}$ are coplanar, and can be resolved into x and y components, with x = 0 as the interface and y = 0 as the normal. θ_i , θ_r and θ_t are as shown in Fig. 1.

By the law of reflection,

$$\theta_i \quad = \quad \theta_r \tag{13}$$

Further, since the incident and reflected waves are in the same medium, travel at the same speed and have the same wavelength,

$$k_i \quad = \quad k_r \tag{14}$$

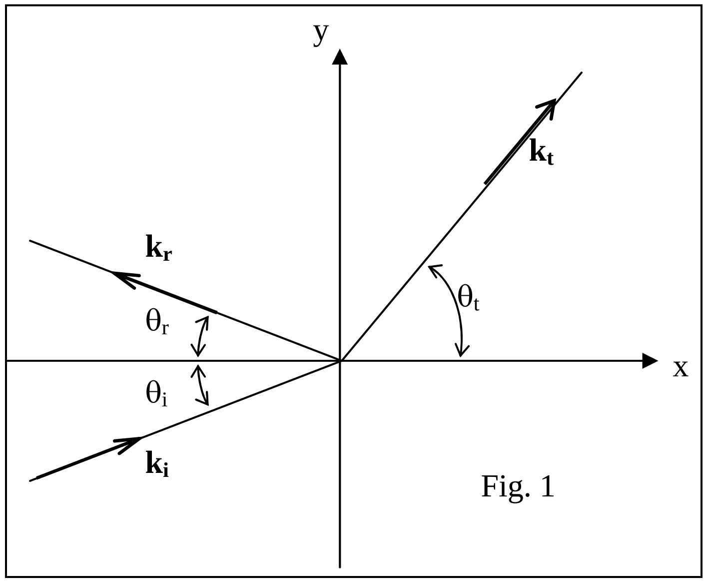


Fig. 1

Resolving equation (12) and substituting (13) and (14):

$$A_i k_i \cos\theta_i \quad - \quad A_r k_i \cos\theta_i \quad = \quad A_t k_t \cos\theta_t \tag{15}$$

$$A_i k_i \sin\theta_i \quad + \quad A_r k_i \sin\theta_i \quad = \quad A_t k_t \sin\theta_t \tag{16}$$

Further, from the relation

$$k_i \eta_t \quad = \quad k_t \eta_i \tag{17}$$

Snell's law gives

$$k_t \sin\theta_t \quad = \quad k_i \sin\theta_i \tag{18}$$

Eliminating $\sin\theta_t$ from Equation (16) retrieves an equation identical to Equation (6).
Substituting $k_i\eta_t = k_t\eta_i$ and A_t from equation (6) into equation (15):

$$\eta_i \cos\theta_i \,(A_i - A_r) \quad = \quad (A_i + A_r)\, \eta_t \cos\theta_t \tag{19}$$

$$\frac{A_r}{A_i} \quad = \quad \frac{\eta_i \cos\theta_i - \eta_t \cos\theta_t}{n_i \cos\theta_i + \eta_t \cos\theta_t} \tag{20}$$

Again from Snell's Law

$$\eta_t \cos\theta_t \quad = \quad \eta_t \sqrt{1-\sin^2\theta_t} \quad = \quad \sqrt{\eta_t^2 - \eta_i^2 \sin^2\theta_i}$$

$$\frac{A_r}{A_i} \quad = \quad \frac{\eta_i \cos\theta_i - \sqrt{\eta_t^{\,2} - \eta_i^{\,2}\sin^2\theta_i}}{n_i \cos\theta_i + \sqrt{\eta_t^{\,2} - \eta_i^{\,2}\sin^2\theta_i}} \qquad \text{(Reflection coefficient)} \tag{21}$$

The corresponding amplitude ratio of the transmitted to incident wave is

$$\frac{A_t}{A_i} \quad = \quad \frac{2\eta_i \cos\theta_i}{n_i \cos\theta_i + \sqrt{\eta_t^{\,2} - \eta_i^{\,2}\sin^2\theta_i}} \qquad \text{(Transmission coefficient)} \tag{22}$$

Discussion

Fig. 2. Plot for $\eta_i < \eta_t$ ($\eta_i = 1$, $\eta_t = 2$)
(External reflection)

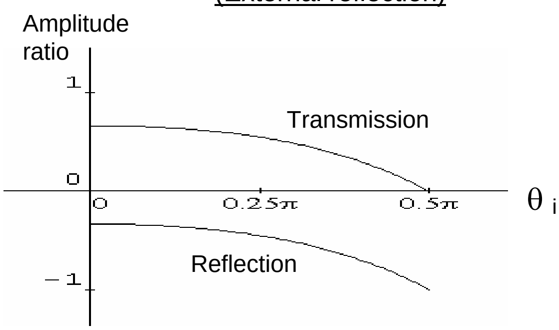
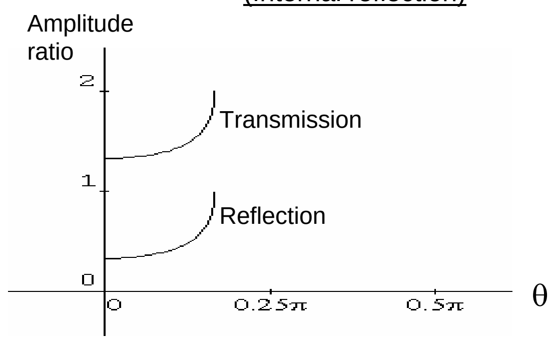


Fig. 3. Plot for $\eta_i > \eta_t$ ($\eta_i = 2$, $\eta_t = 1$)
(Internal reflection)



It may appear odd at first sight that the amplitude of the transmitted wave for internal reflection is greater than the incident amplitude (Fig. 3). However, on closer examination, energy is still conserved. Consider the instantaneous rate of kinetic energy propagation:

$$\frac{d(K.E.)}{dt} = \frac{1}{2}\frac{dm}{dt}\left(\frac{\partial\phi}{\partial t}\right)^2 \qquad (23)$$

The average power is the average rate at which kinetic energy and elastic potential energy are propagated by the wave. The average rate of potential energy transmission is the same as that for kinetic energy transmission. Making the necessary derivative evaluations and substitutions for the appropriate quantities in this case,

$$\frac{P_1}{P_2} = \frac{C_1 v_2 w_1^{\,2} A_1^{\,2}}{C_2 v_1 w_2^{\,2} A_2^{\,2}} \qquad (24)$$

Where P is the power, v the propagation velocity, w the frequency, A the amplitude, and C the cross sectional area, clearer seen in Fig. 4 below.

From Fig. 4, it is clear that
$C_i \sec\theta_i = C_r \sec\theta_r = C_t \sec\theta_t$.

Terming the fraction of transmitted power transmittance and the fraction of reflected power reflectance, substituting the amplitude ratios earlier derived and eliminating θ_t using Snell’s law, we obtain expressions for transmittance and reflectance.

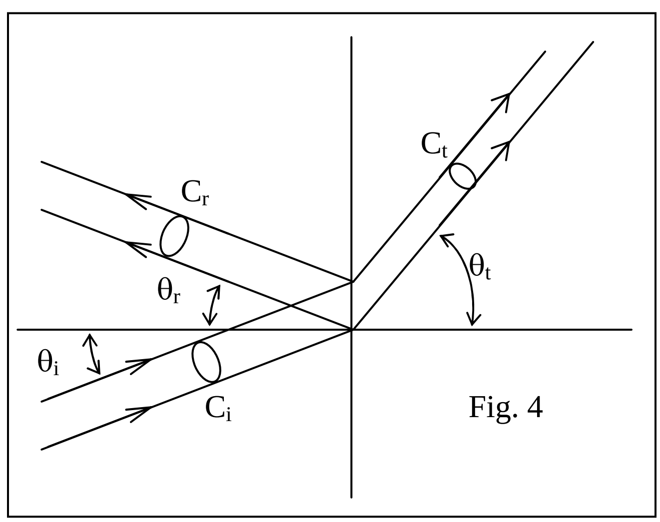


Fig. 4

$$\frac{P_t}{P_i} = \frac{\sqrt{\eta_t^{\,2} - \eta_i^{\,2}\sin^2\theta_i}}{\eta_i\cos\theta_i}\left(\frac{2\eta_i\cos\theta_i}{n_i\cos\theta_i + \sqrt{\eta_t^{\,2} - \eta_i^{\,2}\sin^2\theta_i}}\right)^2 \qquad \text{(transmittance)} \qquad (25)$$

$$\frac{P_r}{P_i} = \left(\frac{A_r}{A_i}\right)^2 = \left(\frac{\eta_i\cos\theta_i - \sqrt{\eta_t^{\,2} - \eta_i^{\,2}\sin^2\theta_i}}{n_i\cos\theta_i + \sqrt{\eta_t^{\,2} - \eta_i^{\,2}\sin^2\theta_i}}\right)^2 \qquad \text{(reflectance)} \qquad (26)$$

It is easily verified that $P_t/P_i + P_r/P_i = 1$, i.e. conservation of energy.

Fig. 5. Plot for $\eta_i < \eta_t$ ($\eta_i = 1$, $\eta_t = 2$) (External reflection)

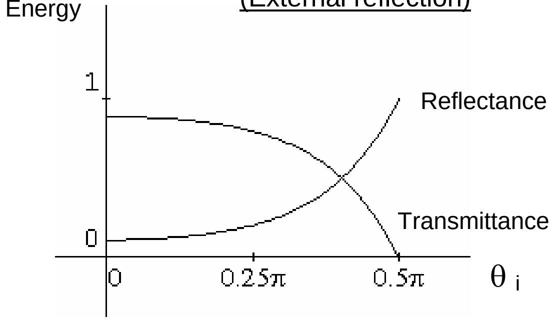
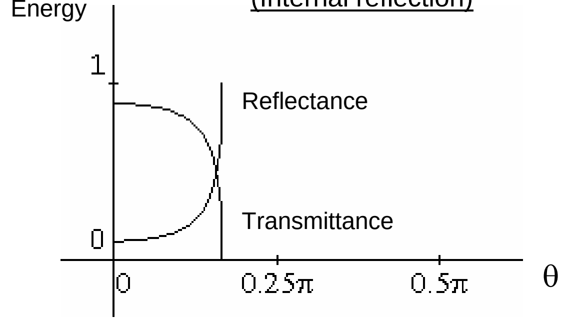


Fig. 6. Plot for $\eta_i > \eta_t$ ($\eta_i = 2$, $\eta_t = 1$) (Internal reflection)



Comparison with Fresnel's equation and its derivation

The results for the amplitude ratios are reminiscent of Fresnel's equations for linearly polarised light with electric field perpendicular to the plane of incidence (and hence tangential to the interface) though the boundary conditions in the derivation of Fresnel's equations are obtained from Maxwell's equations. The boundary condition for the continuity of displacement here is analogous with the continuity of tangential components of electric field at the interface for the case of light. The analogy for continuity of gradient, is however less obvious, though the identical results obtained suggest this. Thus this should be verifiable from Maxwell's equations, which is done below, keeping in mind that the electric fields in consideration are perpendicular to the plane of incidence and tangential to the interface.

Fig. 7 shows an imaginary cylinder.
From Faraday's law of induction,

$$\oint \mathbf{E}\cdot d\mathbf{s} = \frac{d\Phi}{dt}$$

As the "caps" of the cylinder approach the interface, since electric fields tangential to the interface are continuous,

$\oint (\mathbf{E_i}+\mathbf{E_r})\cdot d\mathbf{s}$ approaches

$\oint (\mathbf{E_t})\cdot d\mathbf{s}$, and thus

$\dfrac{d(\Phi_i+\Phi_r)}{dt}$ approaches $\dfrac{d\Phi_t}{dt}$.

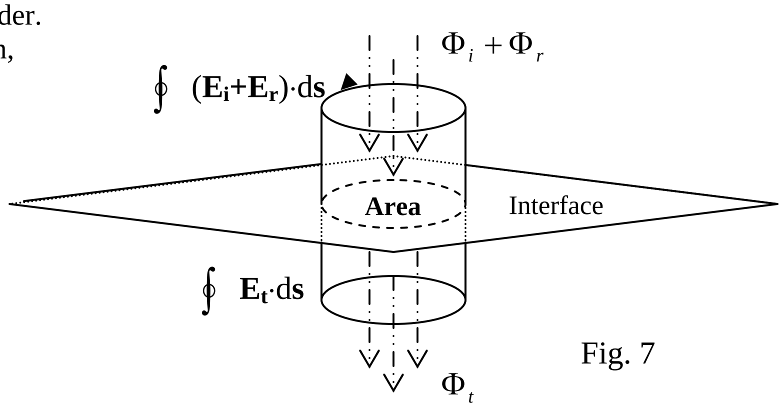


Fig. 7

$\Phi = \mathbf{B} \cdot \mathbf{A} = B\,A \sin\theta$, in this case. Writing down the equation for continuity of rate of change of magnetic flux and then cancelling area A,

$$\frac{dB_i}{dt}\sin\theta_i + \frac{dB_r}{dt}\sin\theta_r = \frac{dB_t}{dt}\sin\theta_t \qquad (27)$$

Snell's law in equation (18) changes the terms in θ to terms in k, the magnitude of the propagation vector. Changing to partial derivative notation,

$$\frac{1}{\mathrm{k_i}}\frac{\partial B_i}{\partial t} + \frac{1}{\mathrm{k_r}}\frac{\partial B_r}{\partial t} = \frac{1}{\mathrm{k_t}}\frac{\partial B_t}{\partial t} \qquad (28)$$

But it can also be worked out from Maxwell's equations that for a travelling electromagnetic wave,

$$\frac{\partial E}{\partial k} = -\frac{\partial B}{\partial t} \qquad (29)$$

where k is the axis of propagation of that electromagnetic wave (which is not the same for the 3 waves in consideration). Now ∇E is in the direction of propagation of that wave, since this is the steepest gradient. Thus from equation (29) and equation (28) it can be deduced that

$$\nabla E_i + \nabla E_r = \nabla E_t \qquad (30)$$

Hence the initial assumption is verified. To conclude, the analogy between spatial displacement and electric field more meaningfully suggests the (otherwise intuitively quite meaningless) continuity of gradient of tangential components of electric field at the interface, which can be verified from Maxwell's equations. It should also be noted that the results cannot compare with that for light polarised with magnetic field perpendicular to the plane of incidence (or a linear combination of both), for which there is another Fresnel equation.

Acknowledgements

Tan Liangzheng is thanked for assistance.

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