LIE GROUPS AND THEIR COSETS

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ABSTRACT

This paper examines elementary Lie groups and their geometrical significance. A brief introduction to Lie groups and their applications is included, following which the basic Lie groups are introduced without going into excess detail about their structure. The relationships between these groups is then examined, especially where certain groups are subgroups of others, in both general and specific cases. In particular, the isomorphic relationship between $SO(2)$ and $U(1)$ is highlighted, as is the homomorphic relationship between $SO(3)$ and $SU(2)$. Following that, we examine certain cosets of these Lie groups and their geometrical representations, which are the focus of this paper. In the conclusion we summarise the main results and discuss their applications in physics.

INTRODUCTION

The study of groups is essentially the study of symmetry [1,2]. In particular, Lie groups represent a class of groups that are continuous rather than discrete, thus having infinitely many elements. Subsequently, they can actually be identified as manifolds, for example, $U(1)$ is a circle while $SU(2)$ is a 3-sphere. The vector space $\Re^n$ may also be considered as an n-dimensional Lie group. Lie groups are highly useful in mathematics and in physics, particularly in the description of electromagnetism and other fundamental forces relating to elementary particles. As such, they have become increasingly important especially in modern theoretical physics.

The rest of the paper explores some basic Lie groups and homomorphisms between certain Lie groups. The results are then applied to the geometrical representations of certain groups along with their cosets. The analysis of quotient groups described by G/H is of particular interest, since they also describe manifolds. Such analysis is often simplier than that of the group G, and can also provide clues about the nature of G itself.

BASIC LIE GROUPS

We will first consider groups that preserve length [3]. For example, it can be shown that for a length-preserving transformation matrix, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acting on a vector $\begin{pmatrix} x \\ y \end{pmatrix}$, its determinant, i.e. $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ is equal to ± 1. An example of this is $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, whose determinant is 1.

From this, we can define $O(n) = \left\{ A_{n\times n} : A^T A = I \right\}$ as the orthogonal groups of order n. These groups represent ordinary spatial rotations and reflections in n-dimensional space. We can also similarly define $SO(n) = \left\{ A_{n\times n} : A^T A = I,\ \det A = 1 \right\}$ as the special orthogonal groups, and by virtue of their definition, $SO(n)$ is a subgroup of $O(n)$. It can also be shown that $SO(n)$ includes all rotations but not reflections.

The dimension of a continuous Lie group is defined as the number of independent parameters required for its expression. Finding the dimension of a group allows us to check whether a

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particular relation between different groups is possible. It can be shown that the dimension of $O(n)$ and $SO(n)$ is $\frac{1}{2}n(n-1)$.

In extending our analysis to complex numbers, we can define the unitary group $U(n) = \{A_{n\times n} : A^\dagger A = I\}$. Consider $A \in U(n)$. Note that $\det A = e^{i\theta}$, $\theta \in \Re$ from $|\det A|^2 = 1$ based on the definition of $U(n)$. $U(n)$ thus describes transformations in n-dimensional complex space. We continue to explore the topological space represented by some specific unitary groups. Let $A = (a) \in U(1)$. By considering its determinant, it can be shown that $a = e^{i\theta}$, $\theta \in [0, 2\pi)$. It is then obvious that $U(1)$ can be represented by a circle with radius 1, otherwise known as S^1.

We can further restrict $U(n)$ by defining the special unitary group $SU(n) = \{A_{n\times n} : A^\dagger A = I,\ \det A = 1\}$. From the definition, for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(2)$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, four equations linking the variables a, b, c and d can be derived, and these can be solved by letting $c = -b^*$, $d = a^*$. These four equations then can be reduced to the single equation $|a|^2 + |b|^2 = 1$, and thus $|\mathrm{Re}(a)|^2 + |\mathrm{Im}(a)|^2 + |\mathrm{Re}(b)|^2 + |\mathrm{Im}(b)|^2 = 1$. $SU(2)$ can thus be considered as a 4-dimensional hypersphere, or an S^3. It should also be noted that $SU(n) \subset U(n)$, being the special case where $\theta = 0$ and thus $\det A = 1$.

We note that the dimension of $U(n)$ is n^2, while that of $SU(n)$, having one more limiting equation in $\det A = 1$, is $n^2 - 1$.

Below is a table to summarise the above groups [4]:

Lie Group	**Matrix Condition**	**Dimension**
$O(n)$	$O(n) = \{A_{n\times n} : A^T A = I\}$	$\frac{1}{2}n(n-1)$
$SO(n)$	$SO(n) = \{A_{n\times n} : A^T A = I,\ \det A = 1\}$	$\frac{1}{2}n(n-1)$
$U(n)$	$U(n) = \{A_{n\times n} : A^\dagger A = I\}$	n^2
$SU(n)$	$SU(n) = \{A_{n\times n} : A^\dagger A = I,\ \det A = 1\}$	$n^2 - 1$

LIE SUBGROUPS

We now look at the relationships between Lie groups, examining how some of them are subgroups of others. Let us consider if $U(n) \subset SU(n+1)$. A check on the dimension reveals that such a relationship is possible. We let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(2)$, and $B \in SU(3)$. If we denote B as $\begin{pmatrix} x & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix}$, then $x \det A = 1$ and thus $x = \frac{1}{\det A}$. From element $A \in U(2)$ we can derive an element $B \in SU(3)$, thus $U(2) \subset SU(3)$ and it can be generalized to show that $U(n) \subset SU(n+1)$. By extension, the result $SO(n) \subset SU(n) \subset U(n) \subset SU(n+1)$ can be shown.

Geometrically, these groups have interesting implications. If we embed $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ in an $SO(3)$ matrix in the form $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$, then by acting this on a point with coordinates (x, y, z):

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y\cos\theta + z\sin\theta \\ -y\sin\theta + z\cos\theta \end{pmatrix}$$

it can be seen that such a matrix represents a rotation of the point about the x-axis by a certain procedure involving the parameter θ. Similarly, with different embeddings, we can deduce matrices for rotations about the y- and z-axes. A multiplication of these three matrices involving parameters $\theta_1, \theta_2, \theta_3$ would thus give the general form of $SO(3)$, representing all sets of rotations possible about the x-, y- and z-axes in $\Re^3$.

ISOMORPHISM BETWEEN $SO(2)$ & $U(1)$

Consider $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in SO(2)$ acting on $\begin{pmatrix} x \\ y \end{pmatrix}$:

$$\begin{aligned} \begin{pmatrix} x' \\ y' \end{pmatrix} &= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \begin{pmatrix} x\cos\theta + y\sin\theta \\ -x\sin\theta + y\cos\theta \end{pmatrix} \end{aligned}$$

Now consider $e^{-i\theta} \in U(1)$, $\theta \in [0, 2\pi)$, acting on (x,y) in the complex plane:

$$\begin{aligned} e^{-i\theta}(x + iy) &= (\cos\theta - i\sin\theta)(x + iy) \\ &= (x\cos\theta + y\sin\theta) + i(y\cos\theta - x\sin\theta) \\ &= x' + iy' \end{aligned}$$

It is thus clear that $U(1)$ and $SO(2)$ are isomorphic with both representing rotation in a single x-y plane. The distinguishing feature is that while $SO(2)$ represents rotations in vector form, $U(1)$ represents them in a complex form.

HOMOMORPHISM BETWEEN $SU(2)$ & $SO(3)$

We explore the relationship between $SU(2)$ and $SO(3)$ following the treatment in [5]. Consider a spinor (ψ, ϕ) defined on a point M on the surface of a unit sphere in $\Re^3$ having coordinates (x,y,z). We can then represent x, y and z as follows, $x = \psi\phi^* + \psi^*\phi,\ y = i(\psi\phi^* - \psi^*\phi),\ z = \psi\psi^* - \phi\phi^*$. A simple check will reveal that this satisfies $x^2 + y^2 + z^2 = 1$, and thus such a substitution will satisfy the stipulated requirement for M. We then define the transformation U such that $U\begin{pmatrix} \psi \\ \phi \end{pmatrix} = \begin{pmatrix} \psi' \\ \phi' \end{pmatrix}$ where $\psi' = a\psi + b\phi,\ \phi' = -b^*\psi + a^*\phi$ and a and b are complex parameters satisfying $aa^* + bb^* = 1$. It can thus be shown that $U = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$ has the standard form of an element of $SU(2)$. The transformation U takes a point M to another point M' of the same sphere. The coordinates of M' can be obtained by substituting in the values of ψ', ϕ' into the expressions for x, y and z. Simplifying, we get:

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \frac{1}{2}[a^2 + (a^*)^2 - b^2 - (b^*)^2]x + \frac{1}{2}i[(a^*)^2 - a^2 - b^2 + (b^*)^2]y - (ab + a^*b^*)z \\ \frac{1}{2}i[a^2 - (a^*)^2 + (b^*)^2 - b^2]x + \frac{1}{2}[a^2 + (a^*)^2 + b^2 + (b^*)^2]y + i(a^*b^* - ab)z \\ (a^*b + ab^*)x + i(a^*b - ab^*)y + (|a|^2 - |b|^2)z \end{pmatrix}$$

The coefficients of x, y and z in the above are all real, and thus the transformation is indeed a rotation in $\Re^3$. This suggests a homomorphic relationship between $SU(2)$ and $SO(3)$, which describes rotations in $\Re^3$. Moreover, substituting a with $-a$ and b with $-b$, will show that the transformations U and $-U$ correspond to the same rotation in $\Re^3$. We also note that U maps $SU(2)$ onto $SO(3)$, and thus $SU(2) \cong SO(3)$ in the ratio 2:1 [6].

We have mentioned above that $SU(2)$ can be geometrically represented as a 3-sphere. With the above relationship, $SO(3)$ can thus be represented as a 3-sphere with antipodal points identified, i.e. RP_3, a real projective space in 4 dimensions [7].

COSETS OF LIE GROUPS

We start this section with a few definitions [8]. For $H \subset G$ and $g \in G$ we define the left coset of H with coset representative g as $gH = \{gh : h \in H\}$. Similarly, a right coset of H is given by $Hg = \{hg : h \in H\}$. Also, H is a normal subgroup of G if $gHg^{-1} = H$ for all $g \in G$. It can then be shown that the set of cosets of H forms a group under the operation on the set of subsets of G induced by the group operation of G. This group of cosets is thus called the coset space, denoted by G/H. We note that G/H is a group, known as a quotient group, if and only if H is a normal subgroup. This is from the definition of a normal subgroup, where the multiplication of two cosets is closed, as shown by $gHg'H = gg'HH = gg'H$. Below we consider a few of these cosets of Lie groups.

Coset space $\frac{\mathbf{O(n+1)}}{\mathbf{O(n)}}$:

This coset space can best be derived using geometrical methods. We consider elements of $O(n+1)$ which are not in $O(n)$. These are in fact those transformations which involve the $(n+1)$-th axis. These include both the rotations, represented by matrices with determinant of 1, and the reflections, represented by matrices with determinant of -1. When such length-preserving transformations act on the point $(1, 0, 0, \dots)$ in $\Re^{n+1}$, the locus is in effect a sphere in $n+1$ dimensions, otherwise known as S^n. The plane where the $(n+1)$-th coordinate is zero is also included as, by definition, the identity element is included in any group. As such, $\frac{O(n+1)}{O(n)} \cong S^n$.

Coset space $\frac{\mathbf{SO(n+1)}}{\mathbf{SO(n)}}$:

By restricting the above analysis of $\frac{O(n+1)}{O(n)}$ to that of $SO(n+1)$ and $SO(n)$, it is clear that a similar conclusion can be drawn. As such, the result $\frac{SO(n+1)}{SO(n)} \cong S^n$ is proven.

Coset space $\frac{\mathbf{O(n)}}{\mathbf{SO(n)}}$:

Let $G = O(n)$ and $H = SO(n)$ [9]. It is clear that H is a normal subgroup of G since $\det ghg^{-1} = \det h = 1$. As such, $\frac{O(n)}{SO(n)}$ must be a group. We examine the case where $n = 3$. Keeping in mind that $\det g = \pm 1$ and $\det h = 1$, if $g \notin H$, then $g = Ph$ where $h \in H$ and $P = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}$. Thus $gH = PH$, and thus the cosets obtained are H and PH. We consider the group multiplication table for these cosets:

	H	PH
H	H	PH
PH	PH	H

This is clearly isomorphic to $\mathrm{Z}_2 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$. As such, $\frac{O(3)}{SO(3)} \cong \mathrm{Z}_2$. Further analysis will show that by representing P as any diagonal matrix with an odd number of entries –1 and the others being 1 will prove that $\frac{O(n)}{SO(n)} \cong \mathrm{Z}_2$ in general.

Quotient group $\frac{\mathbf{SO(n+1)}}{\mathbf{O(n)}}$:

We quote the above results that $\frac{O(n+1)}{O(n)} \cong S^n$ and $\frac{O(n+1)}{SO(n+1)} \cong \mathrm{Z}_2$. Thus $\frac{SO(n+1)}{O(n)} \cong \frac{\frac{O(n+1)}{O(n)}}{\frac{O(n+1)}{SO(n+1)}} \cong \frac{S^n}{Z_2}$. This is equivalent to a 2 to 1 mapping of S^n, and by identifying antipodal points in S^n, we can derive RP_n, a real projective space in $n+1$ dimensions.

Coset space $\frac{\mathbf{U(n)}}{\mathbf{SU(n)}}$:

This coset space is clearly a complex version of the above one. Now let $G = U(n)$ and $H = SU(n)$. From above, $\det g = e^{i\theta}$, $\theta \in [0, 2\pi)$. If $g \notin H$, then $\det g \neq 1$ and thus $\theta \neq 0$. However, when $\theta = 0$, $g = \mathrm{I}$ which by definition must be in H. As such, the above condition for the determinant holds. Subsequently, comparing this form against $U(1)$, it can be shown that $\frac{U(n)}{SU(n)} \cong U(1)$. It is interesting to note that $\mathrm{Z}_2 \subset U(1)$, being the special case when $\theta = 0$ and $\theta = \pi$.

Coset space $\frac{\mathbf{SU(2)}}{\mathbf{U(1)}}$:

This is a special case of $\frac{SU(n+1)}{U(n)}$, whose general consideration is beyond the scope of this paper [10]. To find this case when $n = 1$, we start by finding $\begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \in SU(2)$ such that $a \neq e^{i\theta}$, implying $b \neq 0$. Since $|a|^2 < 1$, we let $a = re^{i\theta}, |r| < 1$. From $|\mathrm{Re}(a)|^2 + |\mathrm{Im}(a)|^2 + |\mathrm{Re}(b)|^2 + |\mathrm{Im}(b)|^2 = 1$, we subsequently obtain $r^2 + |\mathrm{Re}(b)|^2 + |\mathrm{Im}(b)|^2 = 1$, $|r| < 1$. We then consider when $r = 1$ and $a = e^{i\theta} = 1$, obtaining the identity which by definition must be an element of both $SU(2)$ and $U(1)$. We now include the points where $r = \pm 1$, and observe that $r^2 + |\mathrm{Re}(b)|^2 + |\mathrm{Im}(b)|^2 = 1$ is obtained; i.e. the defining equation for S^2. We thus conclude that $\frac{SU(2)}{U(1)} \cong S^2$.

CONCLUSION

The above discussion has shown that there are indeed varied relationships between different Lie groups, some with wide and varied implications. The existence of homomorphisms and isomorphisms between different Lie groups helps to simplify and explain the relationships between seemingly unrelated groups, making comprehension of their nature easier.

The Lie groups introduced in this paper are among the most elementary. Yet they have wide-ranging applications in our world [11]. For example, $SO(4)$ is an important group in the construction of a theory of quantum gravity. $SU(2)$ relates to the gauge group in Yang-Mills theory, while $SU(3)$ relates to that in quantum chromodynamics. The consideration of coset spaces of Lie groups is also important in the study of Kaluza-Klein theory, as discussed by Witten [12]. The discussion of exactly how they are related is beyond the scope of this paper, but it is worthy to note that results derived herein can help those interested explore further.

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